COMMON SUPPORTS FOR PAIRS OF SETS

BY

J. B. COLLIER AND M. EDELSTEIN

ABSTRACT

Sufficient conditions are given for the existence of a closed hyperplane which supports each member of a pair of closed and bounded subsets of a Banach space.

Let A and B be nonempty closed bounded subsets of a Banach space X. A closed hyperplane H is said to support the set A at the point p if $p \in A \cap H$ and A is contained in one of the closed half-spaces determined by H. If H supports both A and B, we call H a common support of A and B. In such a case, either H separates A and B, that is, A and B lie in distinct closed half-spaces determined by H, or H does not separate A and B. The former we call a separating common support and the latter a nonseparating common support. Clearly for H to be a nonseparating common support, it is necessary and sufficient that there be a non-zero $f \in X^*$ with the property that (i) f achieves its supremum on both A and B and (ii) that these suprema be equal. It is interesting to note that (i) may fail in c_0 even for the case when A and B are both symmetric convex bodies [2]. Of course (ii) may fail for a variety of geometric reasons all of a more obvious nature.

The purpose of this note is to give sufficient conditions for the existence of common supports for pairs of closed and bounded sets in a special class of Banach spaces. To this end we define property (σ):

A Banach space X is said to have property (σ) if each closed and bounded convex subset of X is the closed convex hull of the set of its strongly exposed points.

Recall that a point p is said to be a strongly exposed point of a set $A \subset X$ if a non-zero $f \in X^*$ exists such that $f(p) = \sup\{f(a) : a \in A\}$ and whenever $\{x_n\}$ is a sequence in A with $f(x_n) \to f(p)$ then $x_n \to p$.

In [1], it was shown that property (σ) holds for every separable dual Banach space as well as $l_1(\Gamma)$ where Γ is an arbitrary index set. Phelps [3] has recently

Received May 3, 1974

shown that a Banach space X has property (σ) if and only if Rieffel's Radon-Nikodym theorem [4] is valid for that space and, in particular, whenever X is a dual space which is weakly compactly generated.

In the sequel, we shall need the following.

LEMMA 1. *Let B_1, B_2 be subsets of a Banach space X and $B = \mathrm{cl}(B_1 + B_2)$ where $B_1 + B_2 = \{x_1 + x_2 / x_1 \in B_1, x_2 \in B_2\}$. If b is a point of B strongly exposed by $f \in X^*$, then there exists a point $b_i \in B_i$, $i = 1, 2$, strongly exposed by f and $b = b_1 + b_2$.*

PROOF. Without loss of generality we may assume $\sup f[B] = \sup f[B_1] = \sup f[B_2] = 0$. The fact that f strongly exposes a point of B is readily seen to be equivalent to the condition that for each $\epsilon > 0$ there is a $\delta > 0$ such that $D(\delta) = \{x \in B / f(x) \geqq -\delta\}$ is of diameter less than ϵ. If $D_i(\delta) = \{x \in B_i / f(x) \geqq -\delta/2\}$ for $i = 1, 2$, then since $D_1(\delta) + D_2(\delta) \subseteq D(\delta)$, each $D_i(\delta)$ is also of diameter less than ϵ. Thus f strongly exposes each B_i and the point $b_i \in B_i$ which is strongly exposed must be the singleton $\cap \{D_i(\delta) / \delta > 0\}$. Since $f(b) = f(b_1) = f(b_2) = 0$ and the b_i are unique with respect to this property, it is clear that $b = b_1 + b_2$.

Recall that the sets A and B are said to be strongly separated by a hyperplane H if $H + x$ separates A and B for each x in some neighborhood of the origin.

The main result on common supports follows.

THEOREM. *Let X be a Banach space with $\dim X \geqq 2$ and having property (σ). Let A_1 and A_2 be strongly separated closed and bounded subsets of X. Then A_1 and A_2 have both a nonseparating common support and a separating common support.*

PROOF. We first show that A_1 and A_2 have a nonseparating common support. Let H be a hyperplane which strongly separates A_1 and A_2. Choose $H_i, i = 1, 2$, to be a translate of H which strongly separates A_i and H and so that $H = \frac{1}{2}(H_1 + H_2)$. Let $C = \overline{\mathrm{co}}(A_1 \cup A_2)$, $B = C \cap H$, and $B_i = C \cap H_i$, $i = 1, 2$. Since $\frac{1}{2}(B_1 + B_2) \subseteq C$ and $\frac{1}{2}(B_1 + B_2) \subseteq H$, we have $\frac{1}{2}\mathrm{cl}(B_1 + B_2) \subseteq B$. The opposite inclusion follows from $H \cap \mathrm{co}(A_1 \cup A_2) \subseteq \frac{1}{2}(B_1 + B_2)$. Thus $B = \frac{1}{2}\mathrm{cl}(B_1 + B_2)$.

Since X has property (σ), there is a point $b \in B$ which is strongly exposed by some $f \in X^*$. Lemma 1 implies that there are points $b_i \in B_i$ which are strongly exposed by f and $b = \frac{1}{2}(b_1 + b_2)$. We may assume without loss of generality that $b = 0$. Let L be the line through b_1 and b_2 and hence through 0 also. Since X is at least two dimensional, $L \neq X$. The subspace spanned by $f^{-1}[0] \cap H$ and L is a closed hyperplane and therefore the kernel of a non-zero $g \in X^*$ with $\sup g[B] = 0$. Observe that g also strongly exposes 0 as a point of B. Since $\sup g[B_i] = 0$, we have $\sup g[A_i] = 0$, $i = 1,2$. Thus it suffices to show that g actually achieves its supremum on both A_1 and A_2.

Let $\{x_m\}$ be a sequence in A_1 such that $g(x_m) \to 0$. For each m, let x'_m be the point of intersection of H with the line segment $[x_m, b_2]$. Clearly $x'_m \in B$ and $g(x'_m) \geqq g(x_m)$; hence $g(x'_m) \to 0$. Since g strongly exposes 0 as a point of $B, x'_m \to 0$. From this and the fact that A_1 is bounded, it follows that $\{x_m\}$ must have a cluster point p in the line L. Hence $p \in A_1$ and $g(p) = 0$. Since the same argument shows that g also attains its supremum on A_2, the kernel of g must be a nonseparating common support for A_1 and A_2.

We next show that A_1 and A_2 have a separating common support. Let H' and H'' be translates of H such that H' strongly separates $A_1 \cup A_2$ and H''. Each ray with its endpoint in A_1 which passes through a point of A_2 intersects H''. Let A'_1 be the closure of the set of all such intersection points in H''. Since A_1 and A_2 are strongly separated, A'_1 is bounded.

Since A_2 and A'_1 are closed bounded sets strongly separated by H', they have a nonseparating common support. Thus we may suppose that there is a non-zero $g' \in X^*$ such that $\sup g'[A_2] = \sup g'[A'_1] = 0$ and that $0 \in A_2$. Moreover, from the preceeding argument we may further assume that there is a line L' through 0 with the property that whenever $\{y'_m\}$ is a sequence in A'_1 for which $g'(y'_m) \to 0$, then $\{y'_m\}$ has a cluster point in L'. Clearly L' is contained in the kernel of g'.

From the definition of A'_1 it follows that $\inf g'[A_1] = 0$. Let $\{y_m\}$ be a sequence in A_1 for which $g'(y_m) \to 0$. The ray with endpoint y_m which passes through 0 intersects A'_1 in a point y'_m. Clearly $g'(y'_m) \to 0$, hence $\{y'_m\}$ has a cluster point in L'. Since $\{y'_m\}$ has a cluster point in $L', \{y_m\}$ has a cluster point q in L'. Clearly $q \in A_1$ and $g'(q) = \inf g'[A_1] = 0$. Thus the kernel of g' is a separating common support for A_1 and A_2.

References

1. J. Collier and M. Edelstein, *On strongly exposed points and Fréchet differentiability*, Israel J. Math. 17 (1974), 66–68.

2. M. Edelstein and A. C. Thompson, *Some results on nearest points and support properties of convex sets in c_0*, Pacific J. Math. 40 (1972), 553–560.

3. R. R. Phelps, *Dentability and extreme points in Banach spaces*, J. Funct. Anal. **16** (1974), 78–90.

4. M. A. Rieffel, *Dentable subsets of Banach spaces, with applications to a Radon-Nikodym theorem*, Proc. Conf. Functional Analysis, Thompson Book Co., Washington, D. C. 1967, 71–77.

UNIVERSITY OF SOUTHERN CALIFORNIA,
LOS ANGELES, CALIFORNIA, U.S.A.

DALHOUSIE UNIVERSITY,
HALIFAX, NOVA SCOTIA, CANADA